A Sampling Theorem for Bilevel Polygons Using E-Splines

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Abstract

In this paper we present a novel approach for sampling and reconstructing any K-sided convex and bilevel polygon with the use of exponential splines (E-splines) [1]. The Fourier transform of bilevel polygons, Radon transform and the projection-slice theorem are all utilized to reconstruct sampled bilevel polygons. It will be shown that with K+1 projections we are able to perfectly reconstruct a K-sided bilevel polygon from its samples using E-splines as the sampling kernel. By projections we mean line integrals at arbitrary angles $\tan^{-1}(\frac{n}{m})$, where m and n are the indices of the samples. We will also show that the minimum E-spline order required for perfect reconstruction is $N = p.(2K-2)$ where $p = max(m,n)$ needed in order to produce at least K+1 projections.

1 Introduction

Sampling theory plays a fundamental role in modern signal processing and communications. We all know that signals with bandlimited bandwidth can be sampled and reconstructed perfectly with Shannon's famous sampling theorem. Recently, it was shown [2, 3] that it is possible to sample and perfectly reconstruct some classes of non-bandlimited signals. Signals that can be reconstructed using this framework are called signals with Finite Rate of Innovation (FRI) as they can be completely defined by a finite number of parameters.

Reconstruction of bilevel polygons with different sampling kernels have been looked at in [2, 3, 4, 5], however sampling methods for bilevel polygons with exponential splines have not been considered yet. Maravic et al [5] considered sampling and perfectly reconstructing bilevel polygons using the Sinc and Gaussian sampling kernels. Shukla et al [4] proposed an algorithm, from the theory of complex moments, on sampling bilevel polygons with B-splines as the sampling kernel. In [3], it was shown that E-splines, another important family of kernels, can be used as the sampling kernel to sample 1-D FRI signals. However, thus far, E-splines have not been considered to sample polygonal images or 2-D FRI signals in general.

This paper is organised as follows: In Section II we will briefly discuss the sampling setup needed for sampling 2-D FRI signals. In Section III, the Fourier transform representation of bilevel polygons and the projection slice theorem and their use for retrieving unknown parameters of the transform are explained. Finally in section IV, we will introduce our novel approach for sampling bilevel polygons using E-splines.

2 2-D Sampling Setup For FRI Signals

A general 2-D sampling setup for FRI signals is shown in Figure 1. Here, $g(x,y)$ represents the input signal, $\varphi(x,y)$ the sampling kernel, $s_{j,k}$ the samples and Tx, Ty the sampling intervals.

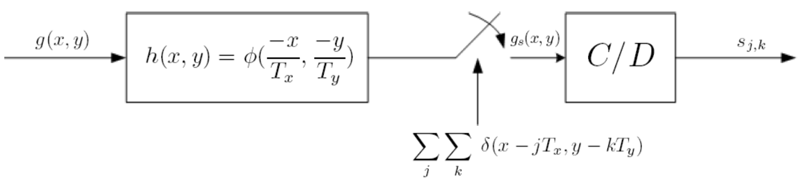

Figure 1: 2-D sampling setup

From the setup shown in Figure 1, the samples $s_{j,k}$ are given by:

$$s_{j,k} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\ \phi(\frac{x}{T_x} - j, \frac{y}{T_y} - k)\ dx\ dy \tag{1}$$

where the kernel $\varphi(x,y)$ is the time reversed version of the filter response. $\varphi(x,y)$ can easily be produced by the tensor product between $\varphi(x)$ and $\varphi(y)$, that is $\varphi(x,y) = \varphi(x) \otimes \varphi(y)$. As mentioned before, $\varphi(x,y)$ is chosen to be an exponential reproducing kernel. Exponential spline kernels (developed by Unser et al [1]) with their shifted versions, can reproduce real or complex exponentials. That is, in 2-D form, any kernel satisfying:

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} c_{j,k}^{m,n} \varphi(x-j, y-k) = e^{\alpha_m x} e^{\beta_n y} \tag{2}$$

is an E-spline for a proper choice of coefficients $c_{j,k}^{m,n}$ which can be found numerically [3]. Here, $m = 0, 1, \ldots, M$, $n = 0, 1, \ldots, N$, $\alpha_m = \alpha_0 + m\lambda_1$ and $\beta_n = \beta_0 + n\lambda_2$.

Before going any further, let us introduce an interesting property here: If we call $\tau_{m,n}$ to be:

$$\tau_{m,n} = \sum_j \sum_k\ c_{j,k}^{m,n}\ s_{j,k} \tag{3}$$

then by expanding $s_{j,k}$ and replacing the above property (assuming $T_x = T_y = 1$ for simplicity), we will obtain the exponential moments of the signal, i.e. :

$$\tau_{m,n} = < g(x,y), \sum_j \sum_k\ c_{j,k}^{m,n}\ \phi(x-j, y-k) > \tag{4}$$

$$= < g(x,y), e^{\alpha_m x} e^{\beta_n y} > \tag{5}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\ e^{\alpha_m x} e^{\beta_n y}\ dx\ dy \tag{6}$$

In the case of purely imaginary exponentials, we will have the discrete Fourier coefficients of the signal $g(x,y)$, that is:

$$\tau_{m,n} = G[\alpha_m, \beta_n] \tag{7}$$

where $G(u,v)$ represents the Fourier transform of the signal $g(x,y)$. This property is very handy when we are reconstructing FRI sampled signals using E-splines. In the next two sections we will illustrate how the use of the Fourier transform representation of bilevel polygons and the projection-slice theorem will help us to sample and perfectly reconstruct any bilevel polygon using E-splines.

3 The Fourier Transform of Bilevel Polygons and the Projection-Slice Theorem

In this section we will show how the Fourier transform of any bilevel polygon is represented and also show how by utilizing the relationship between the Fourier transform and the Radon transform, i.e. the projection-slice theorem, we can retrieve all the unknown parameters of the transform. In [6] S.Lee and R.Mittra derived a general formula for the Fourier transform of any K-sided bilevel polygon where they showed that the Fourier transform of any bilevel polygon is directly related to the location of the polygon's vertices (x_n, y_n) and expressed as:

$$G(u,v) = \sum_{n=1}^{K} e^{j(ux_n + vy_n)}\ \frac{p_{n-1} - p_n}{(u + p_{n-1}v).(u + p_n v)} \tag{8}$$

Here p_n represent the gradients of the polygonal lines. The derived equation closely follows the 2-D harmonic retrieval data model [7], but since the equation has a time-varying amplitude, 2-D harmonic retrieval methods cannot simply be applied to retrieve the locations of the vertices of the polygon. With the use of Radon transform and the projection-slice theorem [8] we obtain an algorithm for retrieving the locations of the vertices of bilevel polygons from their samples. From projection-slice theorem we know that there is a direct relationship between the 2-D Fourier transform and the 1-D Fourier transform of a Radon projection i.e. :

$$G(\omega\cos(\theta), \omega\sin(\theta)) = \hat{R}_g(\omega, \theta) \tag{9}$$

where $G(u, v)$ is the Fourier transform of $g(x, y)$ and $\hat{R}(w, \theta)$ is the 1-D Fourier transform of the Radon transform of $g(x, y)$. With the help of this mapping, we can transform the Fourier coefficients of bilevel polygons, obtained from E-spline sampling kernel (see equation (3)), to the Radon domain, as follows:

$$\hat{R}_g(\omega, \theta) \times \omega^2 = \sum_{n=1}^{N} a_n \times e^{j\omega(cos(\theta)x_n + sin(\theta)y_n)} \tag{10}$$

where a_n is : $\frac{p_{n-1}-p_n}{(cos(\theta)+p_{n-1}sin(\theta)).(cos(\theta)+p_n sin(\theta))}$. Let us introduce $S(\omega, \theta) = \hat{R}_g(\omega, \theta) \times \omega^2$ to present the new mapped equation. Thus, the above equation can be rewritten as:

$$S(\omega, \theta) = \sum_{n=1}^{N} a_n \times e^{j\omega(cos(\theta)x_n + sin(\theta)y_n)} \tag{11}$$

At $\omega = 0$, $S(\omega, \theta) = 0$ so the minimum required spline order can be decreased by 1 as the first data sample is always zero. The mapped equation, at different projections, follows the data model used for the 1-D harmonic retrieval data model exactly, that is:

$$G(\omega, \theta) = \sum_{n=1}^{N} a_n.e^{j\omega z_n} = \sum_{n=1}^{N} a_n.(u_n)^{\omega} \tag{12}$$

where a_n is defined as before, $z_n = cos(\theta)x_n + sin(\theta)y_n$ and $u_n = e^{jz_n}$. Thus, by using a 1-D harmonic retrieval method, for example the Prony's method, we can find all the parameters a_n's and z_n's.

4 A Sampling Theorem for Bilevel Polygons Using Exponential Splines

Consider a non-intersecting, convex and bilevel K-sided polygon with vertices at points (x_n, y_n) where $n = 1, 2, \ldots, K$. The described polygon can be uniquely specified by its K vertices and since each vertex can be described by its x_n and y_n locations, then the polygon has a finite rate of innovation equal to 2K. In this section we will present our novel algorithm for reconstructing convex and bilevel polygons, with E-splines as the sampling kernel.

In section III, we showed that it is possible to retrieve all the parameters a_n's and z_n's by using a 1-D harmonic retrieval method. Now by backprojecting z_n's according to their θ we are able to retrieve some information about the polygon's vertices. The question here is that how many projections will definitely guarantee us to perfectly reconstruct the polygon? Let us assume that the function $g(x, y)$ contains K Diracs, then, as Maravic points out in her paper [9], K+1 projections will entirely specify the signal, i.e. points that have K+1 line intersections from the back-projections correspond to the K Diracs. Any K-sided convex and bilevel polygon is completely specified by the location of its K vertices. If we think of the K vertices as Diracs then K+1 projections will guarantee us to perfectly retrieve the vertices of the bilevel polygon. By projections we mean line integrals at arbitrary angles $\tan^{-1}(\frac{n}{m})$, where m and n are the indices of the samples.

To reconstruct a set of K Diracs from its samples, we need at least 2K data points, thus a minimum 2-D spline order of 2K-1 is required. For bilevel polygons, as the first data sample is always zero, we need a minimum 2-D spline order of 2K-2 at each projection angle. Assuming that the input signal is sampled at a rate $T_x = T_y = T$ with an E-spline order of 2K-2, then 3 immediate projections will be available at the angles 0, 90 and 45 degrees. Since K≥3, more projections will be needed, thus a higher spline order is necessary for the retrieval of all the parameters z_n. The next immediate angles are at $tan^{-1}(2)$ and $tan^{-1}(\frac{1}{2})$, therefore for K = 3 and 4 for example, a minimum spline order of 2(2K-2) = 8 and 12 is required respectively. Moreover, the minimum spline order required for the perfect reconstruction of a given K-sided bilevel polygon is $N = p.(2K-2)$ where $p = max(m, n)$ needed in order to produce at least K+1 projections. From the K+1 projections all the set of parameters are retrieved by using Prony's method, and then normalized by dividing them to $\sqrt{(m^2+n^2)}$. Finally all the retrieved and normalized parameters are back-projected according to their projected angle. Points that have K+1 line intersections correspond to the K vertices of the polygon. Figure 2 shows an example of the sampling process where the input signal, the corresponding samples and the reconstructed signal are all shown.

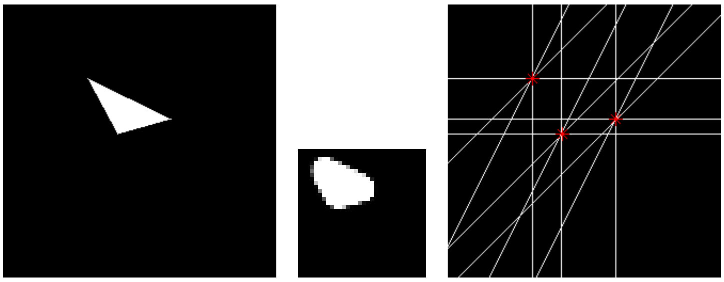

Figure 2: [Not to scale] (a) The original 3-sided polygon in a frame size of 256x256 (b) The 32x32 samples of the input signal (c) The reconstructed vertices with 3+1=4 back-projections, the crosses are the actual vertices of the polygon

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